Math 206A Lecture 28 Notes

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1 Isometric Deformation of Polytopes, Inflating Pillows, and Mylar Balloons

1.1 Isometric deformation of polytopes

Bending of polytopes

Definition 1.1. Let $P \subseteq \mathbb{R}^3$ be a convex polyope. An **isometric deformation** of P is a family $\{S_t, t \in [0,1]\}$ continuous in t such that $S_0 = \partial P$ and $S_t \simeq S_0$ for all t, where $\simeq$ means isometric as metric spaces.

Definition 1.2. $S \simeq S'$ if there exists a homeomorphism $\pi : S \to S'$ such that $|xy|_S = |\pi(x)\pi(y)|_{S'}$.

Example 1.1. Take the unit cube, and punch in one corner to make a cube-shaped indent. If the pinched in cube-shaped indent has side length t, then $\{S_t\}$ is an isometric deformation of the unit cube.

1.2 Inflating pillows

Theorem 1.1. *There are no polyhedral inflated pillows.*

What does this mean? Here is the real theorem.

Theorem 1.2. *For every $P \subseteq \mathbb{R}^3$, there exists a continuous isometric deformation $\{S_T, t \in [0,1]\}$ with $S_0 = \partial P$ such that $\{S_t\}$ is volume increasing.*

Example 1.2. In our punched in cube example, the volume of S_t is $1 - t^3$. This is volume decreasing with t..

What we mean by there are no inflated pillows is that if you take a rectangle and fill it up so that there is as much stuffing as possible, then it cannot be a polyhedron. Why? It it were, we could fill it with more stuffing using a volume-increasing continuous isometric deformation.[1]

We won't prove the full theorem, but here is the main example.

Example 1.3. The main example is when Γ is a unit cube in $\mathbb{R}^3$. On each face, cut out a corner quare of side length t from the cube. We get a surface with 8 holes. Then push out the sides of this cube as much as possible (think of filling the cube with air so the sides puff out). To get a closed surface, we need to fill in the corner holes; we can do this by turning each one into a triangular pyramid shape.

Proposition 1.1. *With the above deformation of the cube,* $\mathrm{Vol}(\hat{S}_t) = 1 + c_1 t + c_2 t^2 + c3t_6$, *where* $c_i \in |R$ *and* $c_1 > 0$.

Proof. S_t is contained in a cube of side length $1 + \alpha t^3$. The length of the sides of the square pyramid must be $\sqrt{2}t$. So $\mathrm{vol}(\hat{S}_t) = (1 + 2\sqrt{2}t)^3 - 12(ct) + O(t^2)$. For small t, $\mathrm{vol}(S_t) = 1 + g\sqrt{2}t + O(t^2)$. □

So the deformation of the cube is volume increasing. This method of deformation is what we want to do in general, but the problem is how to make the corners work out. This is difficult in general but still possible.

1.3 Mylar balloons

When you go to the store, you can buy Mylar balloons. These might say happy birthday on them and have a stick to hold them or something. What these really are is a doubly covered circle, inflated with helium. You can actually calculate the shape and volume of such a shape.[2] A company in Minnesota that produces party balloons actually asked Professor Pak to figure out the shape of party balloons so they could manufacture them more efficiently.[3]

[1] The teabag problem is to figure out what the maximum volume of such a filled up rectangular pillow is.

[2] People have published papers about this. Professor Pak has oulbished a related paper about the shape of a rectangular pillow.

[3] He declined.